

A WEIGHTED ERDŐS–GINZBURG–ZIV THEOREM

DAVID J. GRYNKIEWICZ

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An n -set partition of a sequence S is a collection of n nonempty subsequences of S , pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct so that they can be considered as sets. If S is a sequence of $m+n-1$ elements from a finite abelian group G of order m and exponent k , and if $W = \{w_i\}_{i=1}^n$ is a sequence of integers whose sum is zero modulo k , then there exists a rearranged subsequence $\{b_i\}_{i=1}^n$ of S such that $\sum_{i=1}^n w_i b_i = 0$. This extends the Erdős–Ginzburg–Ziv Theorem, which is the case when $m = n$ and $w_i = 1$ for all i , and confirms a conjecture of Y. Caro. Furthermore, we in part verify a related conjecture of Y. Hamidoune, by showing that if S has an n -set partition $A = A_1, \dots, A_n$ such that $|w_i A_i| = |A_i|$ for all i , then there exists a nontrivial subgroup H of G and an n -set partition $A' = A'_1, \dots, A'_n$ of S such that $H \subseteq \sum_{i=1}^n w_i A'_i$ and $|w_i A'_i| = |A'_i|$ for all i , where $w_i A_i = \{w_i a_i \mid a_i \in A_i\}$.

1. Introduction

Let $(G, +, 0)$ be an abelian group. If $A, B \subseteq G$, then their *sumset*, $A+B$, is the set of all possible pairwise sums, i.e. $\{a+b \mid a \in A, b \in B\}$. A set $A \subseteq G$ is H_a -periodic, if it is the union of H_a -cosets for some subgroup H_a of G (note this definition of periodic differs slightly from the usual definition by allowing H_a to be trivial). A set which is maximally H_a -periodic, with H_a the trivial group, is *aperiodic*, and otherwise we refer to A as *nontrivially periodic*. We say that A is *maximally* H_a -periodic, if A is H_a -periodic, and H_a is the maximal subgroup for which A is periodic; in this case, $H_a = \{x \in$

$G|x+A=A\}$, and H_a is sometimes referred to as the *stabilizer* of A . Note that if A is H_{ka} -periodic, then $A+B$ is also H_{ka} -periodic, whence H_{ka} must be a subgroup of the group H_a for which $A+B$ is maximally H_a -periodic. For a subgroup H , an H -hole of A (where the subgroup H is usually understood) is an element $\alpha \in (A+H) \setminus A$. The *exponent* of G for G finite is the minimal integer k such that $kg=0$ for all $g \in G$. We will regard G as a \mathbb{Z} -module, and will use $\phi_a: G \rightarrow G/H_a$ to denote the natural homomorphism. For $w \in \mathbb{Z}$ and $A \subseteq G$, we let $wA = \{wa_i \mid a_i \in A\}$. If S is a sequence of elements from G , then an n -set partition of S is a collection of n nonempty subsequences of S , pairwise disjoint as sequences, such that every term of S belongs to exactly one of the subsequences, and the terms in each subsequence are all distinct. Thus such subsequences can be considered as sets. We say that S is *zero-sum* if the sum of the terms of S is zero. A *rearranged subsequence* of S is a sequence that under some permutation of terms is a subsequence of S .

In 1961, Erdős, Ginzburg and Ziv proved the following theorem, which can be thought of as a generalization of the pigeonhole principle for m pigeons and two holes [12, 1, 32].

Erdős–Ginzburg–Ziv Theorem (EGZ). *If S is a sequence of $2m-1$ elements from an abelian group G of order m , then S contains an m -term zero-sum subsequence.*

Their theorem spurred the growth of the now developing field of zero-sum Ramsey theory [4, 5, 10, 15, 22, 35], and has been the subject of numerous and varied generalizations [3, 6–9, 13, 14, 16–18, 20, 25, 33, 34]. In the early 1990's, N. Alon proved the following conjecture of Y. Caro in the case $n=m$ with m prime.

Conjecture 1.1. Let $n \geq 2$ and m be integers. Let $W = \{w_i\}_{i=1}^n$ be a sequence of integers such that $\sum_{i=1}^n w_i \equiv 0 \pmod{m}$. Let $S = \{b_i\}_{i=1}^{m+n-1}$ be another sequence of integers. Then there exists a rearranged subsequence $\{b_{j_i}\}_{i=1}^n$ of S such that $\sum_{i=1}^n w_i b_{j_i} \equiv 0 \pmod{m}$.

After communicating with A. Bialostocki and Y. Caro, the proof was soon extended to arbitrary n and m prime, were the status of the problem remained. The conjecture was included a few years later in a survey of Y. Caro on problems in zero-sum combinatorics [10], where a reference was made to the (unpublished) proof of the prime case [2]. Soon after, Y. Hamidoune published a pair of papers where, in the first he proved that an equivalent form of Conjecture 1.1 holds (in a more general abelian group setting) provided each w_i for $i \in \{1, \dots, n-1\}$ was relatively prime to m [26], and in the second he introduced the following conjecture which he verified for $n=m$ [27].

Conjecture 1.2. Let $n \geq 2$ and m be integers. Let S be a sequence of $m + n - 1$ elements from a finite abelian group G of order m , and let $W = \{w_i\}_{i=1}^n$ be a sequence of integers whose sum is zero modulo m . If the multiplicity of every term of S is at most n , and if each w_i for $i \leq n - 1$ is relatively prime to m , then there is a nontrivial subgroup H of G such that for every $h \in H$ there is a rearranged subsequence $\{b_{h_i}\}_{i=1}^n$ of S with $\sum_{i=1}^n w_i b_{h_i} = h$.

No further progress was made on either conjecture until 2003, when W. Gao and X. Jin established [Conjecture 1.1](#) in the case of $m = p^2$, for p prime [19].

The main result of this paper is the following theorem, which completely affirms [Conjecture 1.1](#) for all m and W .

Theorem 1.1. Let m , k and $n \geq 2$ be positive integers. If S is a sequence of $m + n - 1$ elements from a nontrivial abelian group G of order m and exponent k , and if $W = \{w_i\}_{i=1}^n$ is a sequence of integers whose sum is zero modulo k , then there exists a rearranged subsequence $\{b_i\}_{i=1}^n$ of S such that $\sum_{i=1}^n w_i b_i = 0$. Furthermore, if S has an n -set partition $A = A_1, \dots, A_n$ such that $|w_i A_i| = |A_i|$ for all i , then there exists a nontrivial subgroup H of G and an n -set partition $A' = A'_1, \dots, A'_n$ of S with $H \subseteq \sum_{i=1}^n w_i A'_i$ and $|w_i A'_i| = |A'_i|$ for all i .

We note that the example $W = (\overbrace{1, \dots, 1}^{m-2}, 0, 2)$ and $S = (-1, \overbrace{0, \dots, 0}^{m-1}, \overbrace{1, \dots, 1}^{m-1})$ with $G = \mathbb{Z}_m$ cyclic shows that in the above theorem we cannot require $\{b_i\}_{i=1}^n$ to be an actual (including order) subsequence of S . Also, it is easily seen that a sequence S has an n -set partition if and only if its length $|S|$ is at least n and every term of S has multiplicity at most n . Hence, since $|w_i A_i| = |A_i|$ for w_i relatively prime to k (and since both conditions (b) and (c) to be stated at the end of the sentence imply there exists an n -set partition of S with at least one set A_i of cardinality one), it follows that [Theorem 1.1](#) implies [Conjecture 1.2](#) provided any one of the following conditions also holds: (a) w_n is relatively prime to m , or (b) $n \geq m$, or (c) every term of S has multiplicity at most $n - 1$.

We conclude the paper with some remarks concerning two weighted analogs of the Cauchy–Davenport Theorem [11] that generalize [Theorem 1.1](#) provided every w_i is relatively prime to k .

2. Weighted EGZ

We begin by stating the classical theorem of Kneser for sumsets [30, 28, 31, 29, 32, 24]. The case with m prime is known as the Cauchy–Davenport

Theorem [11], and was the original tool used in the proof of [Conjecture 1.1](#) for m prime.

Kneser's Theorem. *Let G be an abelian group, and let A_1, A_2, \dots, A_n be a collection of finite, nonempty subsets of G . If $\sum_{i=1}^n A_i$ is maximally H_a -periodic, then*

$$\left| \sum_{i=1}^n \phi_a(A_i) \right| \geq \sum_{i=1}^n |\phi_a(A_i)| - n + 1.$$

Note that if A is maximally H_a -periodic, then $\phi_a(A)$ is aperiodic. Also, observe that if $A+B$ is maximally H_a -periodic and $\rho = |A+H_a| - |A| + |B+H_a| - |B|$ is the number of holes in A and B , then [Kneser's Theorem](#) implies $|A+B| \geq |A| + |B| - |H_a| + \rho$. Consequently, if either A or B contains a unique element from some H_a -coset, then $|A+B| \geq |A| + |B| - 1$. More generally, if $\rho = \sum_{i=1}^n (|A_i + H_a| - |A_i|)$ is the total number of holes in the A_i , then $|\sum_{i=1}^n A_i| \geq \sum_{i=1}^n |A_i| - (n-1)|H_a| + \rho$.

We will also need the following theorem, which for abelian groups is an easy consequence of [Kneser's Theorem](#) (Theorem 3.2 and Lemma 3.3 in [29]).

Theorem 2.1. *Let A and B be finite, nonempty subsets of a group with $|B| \geq 2$, and let $b \in B$. If $A+B \neq A+(B \setminus \{b\})$, then $|A+B| \geq |A| + |B| - 1$.*

We can now begin the proof of [Theorem 1.1](#), which follows ideas from the proof of a recent composite analog of the Cauchy–Davenport Theorem [23]. In the proof, we will essentially be considering an n -set partition $A = A_1, \dots, A_n$ of S that iteratively maximizes $\sum_{i=1}^n |w_i A_i|$, $|\sum_{i=1}^n w_i A_i|$, and $\sum_{i=1}^n |\phi_a(w_i A_i)|$, where $\sum_{i=1}^n w_i A_i$ is maximally H_a -periodic. With the help of [Kneser's Theorem](#), we will be able to show that we can remove some term b of S from the set partition A leaving the third maximized quantity unaffected. If the second maximized quantity is also preserved, then this will allow us to place the term b back into the n -set partition in such a way as to preserve the first quantity and increase one of the later two quantities, a contradiction, unless the term $\phi_a(b)$ is contained in every set $w_i A_i$, in which case $H_a = \sum_{i=1}^n w_i b + H_a \subseteq \sum_{i=1}^n w_i A_i$ will follow from [Kneser's Theorem](#), completing the proof. On the other hand, if removing the term b from its set $w_j A_j$ would destroy the second maximized quantity, then we will use [Theorem 2.1](#) to show that the set $w_j A_j$ locally adds lots of elements to the sumset $\sum_{i=1}^n w_i A_i$. An extremal argument will then be used to show that either there must be a term of S that can be removed from A while preserving both the later two maximized quantities, or else there will be many sets $w_i A_i$ which locally add lots of elements to $\sum_{i=1}^n w_i A_i$, enough so that we can conclude that the sumset $\sum_{i=1}^n w_i A_i$ has large enough cardinality globally to represent every element of G .

Proof. If there is a term x of S whose multiplicity is at least $n+1$, then S cannot have an n -set partition and [Theorem 1.1](#) follows by choosing $b_i = x$ for all i . Hence we may assume each term of S has multiplicity at most n , whence it easily follows that there exists an n -set partition $A = A_1, \dots, A_n$ of S . Choose A such that $\sum_{i=1}^n |w_i A_i|$ is maximal.

Suppose $|w_j A_j| < |A_j|$ for some index j (so that the conditions from the furthermore part of [Theorem 1.1](#) do not hold), and let $b, b' \in A_j$ with $w_j b = w_j b'$ and $b \neq b'$. If there exists an index r such that $w_r b \notin w_r A_r$, then the n -set partition defined by $A'_j = A_j \setminus \{b\}$, $A'_r = A_r \cup \{b\}$ and $A'_i = A_i$ for $i \neq j, r$, contradicts the maximality of $\sum_{i=1}^n |w_i A_i|$. Thus we may assume $w_i b \in w_i A_i$ for all i . Hence, since $\sum_{i=1}^n w_i \equiv 0 \pmod k$, it follows that $0 = \sum_{i=1}^n w_i b \in \sum_{i=1}^n w_i A_i$, and the proof is complete by an appropriate selection of a term from each A_i . So we may assume $|w_i A_i| = |A_i|$ for all i . Furthermore, assume A is chosen such that $|\sum_{i=1}^n w_i A_i|$ is maximal subject to $|w_i A_i| = |A_i|$ for all i .

If $|\sum_{i=1}^n w_i A_i| \geq m$, then the proof is complete with $H = G$. Hence, since $|S| = m + n - 1$ and since $|w_i A_i| = |A_i|$, it follows that we may assume that

$$(1) \quad \left| \sum_{i=1}^n w_i A_i \right| < \sum_{i=1}^n |w_i A_i| - n + 1,$$

whence from [Kneser's Theorem](#) it follows that $\sum_{i=1}^n w_i A_i \stackrel{\text{def}}{=} X$ is maximally H_a -periodic for some proper, nontrivial subgroup H_a of G . Assume that A was chosen, from among all n -set partitions $A' = A'_1, \dots, A'_n$ of S that satisfy $|w_i A'_i| = |A_i|$ and $\sum_{i=1}^n w_i A'_i = X$, such that $\sum_{i=1}^n |\phi_a(w_i A_i)|$ is maximal.

If every set $w_i A_i$ with $i \geq 2$ contains an element which is the unique element from its H_a -coset in $w_i A_i$, then there are at least $(n-1)(|H_a|-1)$ holes among the sets $w_i A_i$, whence [Kneser's Theorem](#) implies that $|\sum_{i=1}^n w_i A_i| \geq \sum_{i=1}^n |w_i A_i| - (n-1)|H_a| + (n-1)(|H_a|-1) = \sum_{i=1}^n |A_i| - n + 1$, contradicting (1). Therefore we may assume $|\phi_a(w_j A_j)| < |w_j A_j|$ for some index $j \geq 2$.

Let $j \geq 2$ be an index such that $|\phi_a(w_j A_j)| < |w_j A_j|$. Suppose that

$$(2) \quad \sum_{i=1}^j w_i A_i = \sum_{i=1}^{j-1} w_i A_i + w_j (A_j \setminus \{b\}),$$

for some $b \in A_j$ such that $\phi_a(w_j A_j) = \phi_a(w_j (A_j \setminus \{b\}))$. Hence, if there exists an index r such that $\phi_a(w_r b) \notin \phi_a(w_r A_r)$, then the n -set partition defined by $A'_j = A_j \setminus \{b\}$, $A'_r = A_r \cup \{b\}$ and $A'_i = A_i$ for $i \neq j, r$, contradicts the maximality of either $|\sum_{i=1}^n w_i A_i|$ or $\sum_{i=1}^n |\phi_a(w_i A_i)|$. So we may assume $\phi_a(w_i b) \in \phi_a(w_i A_i)$ for all i . Hence, since $\sum_{i=1}^n w_i \equiv 0 \pmod k$, it follows that $0 = \sum_{i=1}^n \phi_a(w_i b) \in \sum_{i=1}^n \phi_a(w_i A_i)$. Thus, since $\sum_{i=1}^n w_i A_i$ is H_a -periodic, it

follows that $H_a \subseteq \sum_{i=1}^n w_i A_i$, and the proof is complete with $H = H_a$. So we may assume that (2) does not hold, whence in view of [Theorem 2.1](#) it follows that

$$(3) \quad \left| \sum_{i=1}^j w_i A_i \right| \geq \left| \sum_{i=1}^{j-1} w_i A_i \right| + |w_j A_j| - 1.$$

Let l , where $2 \leq l \leq n$, be the minimal integer, allowing re-indexing of the $w_i A_i$, such that

$$(4) \quad \left| \sum_{i=1}^j w_i A_i \right| \geq \left| \sum_{i=1}^{j-1} w_i A_i \right| + |w_j A_j| - 1,$$

for all $j \geq l$. From the conclusions of the last two paragraphs, and since by re-indexing we may assume $j = n$ in the previous paragraph, it follows that l exists. Observe that

$$(5) \quad \left| \sum_{i=1}^{l-1} w_i A_i \right| < \sum_{i=1}^{l-1} |w_i A_i| - (l-1) + 1,$$

since otherwise applying (4) iteratively yields $|\sum_{i=1}^n w_i A_i| \geq \sum_{i=1}^n |w_i A_i| - n + 1$, contradicting (1). Hence from [Kneser's Theorem](#) and the maximality of H_a , it follows that $\sum_{i=1}^{l-1} w_i A_i$ is maximally H_{ka} -periodic for some non-trivial subgroup $H_{ka} \leq H_a$. Note that (5) can only hold provided $l-1 \geq 2$. Furthermore, if every set $w_i A_i$ with $2 \leq i \leq l-1$ contains an element which is the unique element from its H_{ka} -coset in $w_i A_i$, then there will be at least $(l-2)(|H_{ka}|-1)$ holes among the sets $w_i A_i$ with $i \leq l-1$, whence [Kneser's Theorem](#) implies that $|\sum_{i=1}^{l-1} w_i A_i| \geq \sum_{i=1}^{l-1} |w_i A_i| - (l-2)|H_{ka}| + (l-2)(|H_{ka}|-1) = \sum_{i=1}^{l-1} |A_i| - (l-1) + 1$, contradicting (5). Therefore there must exist a set A_j with $2 \leq j \leq l-1$, such that $w_j A_j$ does not contain an element which is the unique element from its H_{ka} -coset in $w_j A_j$. Hence, since $H_{ka} \leq H_a$, it follows that $|\phi_a(w_j A_j)| < |w_j A_j|$ for some index j with $2 \leq j \leq l-1$. Thus, since by re-indexing we may assume $j = l-1$, it follows that (3) holds with $j = l-1$, which in view of (4) contradicts the minimality of l . ■

[Theorem 1.1](#) has a much more potent generalization, provided every w_i is relatively prime to k . The following theorem, which implies [Theorem 1.1](#) when every w_i is relatively prime to k , was proved in the case $w_i = 1$ for all i in [23]. However, noting for w_i relatively prime to k that $w_i b \in w_i A_i$ if and only if $b \in A_i$, and allowing for re-indexing of the weights $\{w_i\}_{i=1}^n$, it follows that the proof given in [23] goes through in the more general weighted case with no further modifications other than to insert the weights w_i at appropriate points in the proof.

Theorem 2.2. *Let S be a finite sequence of elements from an abelian group G whose torsion subgroup has exponent k , let $W = \{w_i\}_{i=1}^n$ be a sequence of integers relatively prime to k , let $A = A_1, \dots, A_n$ be an n -set partition of S , and let $a_i \in A_i$ for $i \in \{1, \dots, n\}$. Then there exists an n -set partition $A' = A'_1, \dots, A'_n$ of S such that $\sum_{i=1}^n w_i A'_i$ is H_a -periodic, $\sum_{i=1}^n w_i A_i \subseteq \sum_{i=1}^n w_i A'_i$, $a_i \in A'_i$ for $i \in \{1, \dots, n\}$, and*

$$\left| \sum_{i=1}^n w_i A'_i \right| \geq (E(A', H_a) + (N(A', H_a) - 1)n + 1) |H_a|,$$

where $N(A', H_a) = \frac{1}{|H_a|} |\bigcap_{i=1}^n (A'_i + H_a)|$ and $E(A', H_a) = \sum_{j=1}^n (|A'_j| - |A'_j \cap \bigcap_{i=1}^n (A'_i + H_a)|)$.

Finally, we remark that the proof when $w_i = 1$ for all i of the following corollary to [Theorem 2.2](#) likewise goes through in the more general weighted case (Theorem 2.7 in [21] and Corollary 1 in [23]).

Theorem 2.3. *Let S be a sequence of elements from an abelian group G of order m and exponent k with an n -set partition $P = P_1, \dots, P_n$, let $W = \{w_i\}_{i=1}^n$ be a sequence of integers relatively prime to k , and let p be the smallest prime divisor of m . Then either:*

(i) *there exists an n -set partition $A = A_1, A_2, \dots, A_n$ of S such that:*

$$\left| \sum_{i=1}^n w_i A_i \right| \geq \min \{m, (n+1)p, |S| - n + 1\};$$

furthermore, if $n' \geq \frac{m}{p} - 1$ is an integer such that P has at least $n - n'$ cardinality one sets and if $|S| \geq n + \frac{m}{p} + p - 3$, then we may assume there are at least $n - n'$ cardinality one sets in A , or

(ii) (a) *there exists $\alpha \in G$ and a nontrivial proper subgroup H_a of index a such that all but at most $a - 2$ terms of S are from the coset $\alpha + H_a$; and (b) there exists an n -set partition A_1, A_2, \dots, A_n of the subsequence of S consisting of terms from $\alpha + H_a$ such that $\sum_{i=1}^n w_i A_i = \sum_{i=1}^n w_i \alpha + H_a$.*

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David J. Gryniewicz

Mathematics 253-37

Caltech

Pasadena, CA 91125

USA

diambri@hotmail.com